Modern Linear Algebra

(A Geometric Algebra crash course, Part IV: Transformation of coordinates & Gaussian method of solving a system of linear equations)

Teaching & learning contents according to the modular description of LV 200691.01

- Linear functions, multidimensional linear models, matrix algebra
- Systems of linear equations including methods for solving a system of linear equations and examples in business processes

Most of this will be discussed in the standard language of the rather old-fashioned linear algebra or matrix algebra which can be found in most textbooks of business mathematics or mathematical economics.

But in this part we will again adopt a more modern view: The Gaussian method of solving systems of linear will be interpreted geometrically as a transformation of coordinates.
Repetition: Basics of Geometric Algebra

1 + 3 + 3 + 1 = 2^3 = 8 different base elements exist in three-dimensional space.

One base scalar: 1

Three base vectors: \(\sigma_x, \sigma_y, \sigma_z\)

Three base bivectors: \(\sigma_x\sigma_y, \sigma_y\sigma_z, \sigma_z\sigma_x\)
(sometimes called pseudovectors)

One base trivector: \(\sigma_x\sigma_y\sigma_z\)
(sometimes called pseudoscalar)

Base scalar and base vectors square to one:

\[
1^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1
\]

Base bivectors and base trivector square to minus one:

\[
(\sigma_x\sigma_y)^2 = (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y\sigma_z)^2 = -1
\]
Anti-Commutativity

The order of vectors is important. It encodes information about the orientation of the resulting area elements.

Base vectors anticommute. Thus the product of two base vectors follows Pauli algebra:

\[
\begin{align*}
\sigma_x \sigma_y &= - \sigma_y \sigma_x \\
\sigma_y \sigma_z &= - \sigma_z \sigma_y \\
\sigma_z \sigma_x &= - \sigma_x \sigma_z
\end{align*}
\]
Scalars

Scalars are geometric entities without direction. They can be expressed as a multiple of the base scalar:

\[ k = k \ 1 \]

Vectors

Vectors are oriented line segments. They can be expressed as linear combinations of the base vectors:

\[ r = x \ \sigma_x + y \ \sigma_y + z \ \sigma_z \]

Bivectors

Bivectors are oriented area elements. They can be expressed as linear combinations of the base bivectors:

\[ A = A_{xy} \ \sigma_x \sigma_y + A_{yz} \ \sigma_y \sigma_z + A_{zx} \ \sigma_z \sigma_x \]

Trivectors

Trivectors are oriented volume elements. They can be expressed as a multiple of the base trivector:

\[ V = V_{xyz} \ \sigma_x \sigma_y \sigma_z \]
Geometric Multiplication of Vectors

The product of two vectors consists of a scalar term and a bivector term. They are called inner product (dot product) and outer product (exterior product or wedge product).

\[ a \ b = a \cdot b + a \wedge b \]

The inner product of two vectors is a commutative product as a reversion of the order of two vectors does not change it:

\[ a \cdot b = b \cdot a = \frac{1}{2} (a \ b + b \ a) \]

The outer product of two vectors is an anti-commutative product as a reversion of the order of two vectors changes the sign of the outer product:

\[ a \wedge b = - b \wedge a = \frac{1}{2} (a \ b - b \ a) \]
Systems of Two Linear Equations

\[ a_1 x + b_1 y = d_1 \quad \Rightarrow \quad a x + b y = d \]
\[ a_2 x + b_2 y = d_2 \]

Old column vector picture:

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
\end{bmatrix}
\quad
\begin{bmatrix}
  b_1 \\
  b_2 \\
\end{bmatrix}
\quad
\begin{bmatrix}
  d_1 \\
  d_2 \\
\end{bmatrix}
\]

Modern Geometric Algebra picture:

\[
(a_1 \sigma_x + a_2 \sigma_y) x + (b_1 \sigma_x + b_2 \sigma_y) y = d_1 \sigma_x + d_2 \sigma_y
\]

Solutions:

\[
x = \frac{1}{a \wedge b} (d \wedge b) = (a \wedge b)^{-1} (d \wedge b)
\]
\[
y = \frac{1}{a \wedge b} (a \wedge d) = (a \wedge b)^{-1} (a \wedge d)
\]
Systems of Three Linear Equations

\[a_1 x + b_1 y + c_1 z = d_1\]
\[a_2 x + b_2 y + c_2 z = d_2\] \(\Rightarrow\) \[a x + b y + c z = d\]
\[a_3 x + b_3 y + c_3 z = d_3\]

Old column vector picture:

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix}
\]

Modern Geometric Algebra picture:

\[
(a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z) x + (b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z) y
+ (c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z) z = d_1 \sigma_x + d_2 \sigma_y + d_3 \sigma_z
\]

Solutions:

\[x = (a \wedge b \wedge c)^{-1} (d \wedge b \wedge c)\]
\[y = (a \wedge b \wedge c)^{-1} (a \wedge d \wedge c)\]
\[z = (a \wedge b \wedge c)^{-1} (a \wedge b \wedge d)\]

This is the end of the repetition. More about the basics of Geometric Algebra can be found in the slides of former lessons (see part I, II, & III).
Searching for a Graphical Representation of the Gaussian Method

Let’s start with an example of a rather simple system of two linear equations:

\[4 \, x + 2 \, y = 14\]
\[x + 4 \, y = 14\]

There are several different strategies to solve this system of two linear equations using the method of Gauss.

First strategy:

\[
\begin{array}{ccc|c}
\hline
x & y & d \\
\hline
4 & 2 & 14 \\
1 & 4 & 14 \\
\hline
1 & ? & ? \\
0 & ? & ? \\
\hline
1 & 0 & ? \\
0 & 1 & ? \\
\hline
\end{array}
\]

Second strategy:

\[
\begin{array}{ccc|c}
\hline
x & y & d \\
\hline
4 & 2 & 14 \\
1 & 4 & 14 \\
\hline
? & 0 & ? \\
? & 1 & ? \\
\hline
1 & 0 & ? \\
0 & 1 & ? \\
\hline
\end{array}
\]

⇒ The left-hand side of the augmented matrix is transformed into an identity matrix.
Third strategy:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
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<tbody>
<tr>
<td>4</td>
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<td>14</td>
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<td>4</td>
<td>14</td>
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<tr>
<td>0</td>
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<td>1</td>
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<tr>
<td>0</td>
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<td>?</td>
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<tr>
<td>1</td>
<td>0</td>
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</table>

Fourth strategy:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
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<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>?</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>0</td>
<td>?</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>?</td>
</tr>
</tbody>
</table>

⇒ The left-hand side of the augmented matrix is transformed into a permutation matrix, which exchanges the x- and y-coordinates.

This is an example of the most simple permutation symmetry, called $S_2$ permutation symmetry, as it exchanges only two different entities.

To find out what happens when the Gaussian method is applied, we are looking for diagrams (and equations) which represent these four different strategies graphically (and algebraically).
Searching for a Graphical Representation of the Gaussian Method

In modern linear algebra we transfer all vectors of the system of linear equations

\[
\begin{align*}
4x + 2y &= 14 \\
x + 4y &= 14
\end{align*}
\]

into Pauli vectors:

\[
\begin{align*}
a &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} & \quad \Rightarrow \quad a &= 4 \sigma_x + \sigma_y \\
b &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} & \quad \Rightarrow \quad b &= 2 \sigma_x + 4 \sigma_y \\
d &= \begin{bmatrix} 14 \\ 14 \end{bmatrix} & \quad \Rightarrow \quad d &= 14 \sigma_x + 14 \sigma_y
\end{align*}
\]

They can now be represented in a diagram (see following slide).
The algebraic starting point is:

\[
\begin{align*}
\mathbf{a} & \quad \mathbf{x} + \quad \mathbf{b} \quad \mathbf{y} = \quad \mathbf{d} \\
(4 \sigma_x + \sigma_y) \mathbf{x} + (2 \sigma_x + 4 \sigma_y) \mathbf{y} & = 14 \sigma_x + 14 \sigma_y
\end{align*}
\]
Geometric Analysis of the Starting Point

Algebraic starting point: \( a \times + b \times y = d \)

Problem: How long are the two sides \( a \times \) and \( b \times y \) of the parallelogram compared to the lengths of the coefficient vectors \( a \) and \( b \)?
Geometric Analysis of the Starting Point

Algebraic starting point: \( \mathbf{a} x + \mathbf{b} y = \mathbf{d} \)

Problem: How long are the two sides \( \mathbf{a} x \) and \( \mathbf{b} y \) of the parallelogram?

Solution idea: Coefficient vectors \( \mathbf{a} \) and \( \mathbf{b} \) will become unit vectors of a new coordinate system.
Solution Strategy

Algebraic starting point: \( a \times + b \times y = d \)

Problem: How long are the two sides \( a \times \) and \( b \times y \) of the parallelogram compared to the lengths of the coefficient vectors \( a \) and \( b \)?

If we measure the lengths of both sides \( a \times \) and \( b \times y \) of the parallelogram and compare them with the lengths of the coefficient vectors \( a \) and \( b \), the unknown variables \( x \) and \( y \) will be:

\[
\text{unknown variable} = \frac{\text{length of side of parallelogram}}{\text{length of corresponding coefficient vector}}
\]

\[
\Rightarrow \quad x = \frac{a \times}{a} \quad y = \frac{b \times y}{b} \quad \text{Yes, we are allowed to divide by vectors in Geometric Algebra!}
\]

As the division by vectors is a conceptual problem in standard vector algebra, Gauss decided for another idea:

Solution idea: The axes of the coordinate system will be transformed. Coefficient vectors \( a \) and \( b \) will become unit vectors of the new coordinate system.
Intermediate Step (first strategy)

Change of x-coordinates:

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = 14 \sigma_x + 14 \sigma_y\]

\[e_x = 4 \sigma_x + \sigma_y\]

\[\Rightarrow \sigma_x = \frac{1}{4} e_x - \frac{1}{4} \sigma_y\]

New Gaussian Pauli vector equation:

\[e_x x + \left(\frac{1}{2} e_x + \frac{7}{2} \sigma_y\right) y = \frac{7}{2} e_x + \frac{21}{2} \sigma_y\]
Augmented Matrix of the Intermediate Step (first strategy)

The coefficients of this new Gaussian Pauli vector equation

$$(1 \ e_x + 0 \ \sigma_y) \ x + \left( \frac{1}{2} \ e_x + \frac{7}{2} \ \sigma_y \right) \ y = \frac{7}{2} \ e_x + \frac{21}{2} \ \sigma_y$$

are the elements of the intermediate step of the augmented matrix:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>21</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>?</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>?</td>
</tr>
</tbody>
</table>

In some cases it is helpful to split this intermediate step into sub-steps to understand the strategy of Gauss.
### Augmented Matrix of the Intermediate Step (first strategy)

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<tbody>
<tr>
<td>x</td>
<td>y</td>
<td>d</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

Multiply row one by \( \frac{1}{4} \).

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<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{7}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

Subtract row one from row two.

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<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{7}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{7}{2} )</td>
<td>( \frac{21}{2} )</td>
</tr>
</tbody>
</table>

As intended, coefficient vector \( \mathbf{a} \) now represents a unit vector which lies parallel to the new \( x \)-coordinate axis:

\[ \mathbf{e}_x = \mathbf{a} \]
## Allowed Row Operations to Transform Augmented Matrices

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

Multiply row one by $\frac{1}{4}$.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{7}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

Subtract row one from row two.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{7}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{7}{2}$</td>
<td>$\frac{21}{2}$</td>
</tr>
</tbody>
</table>

The following row operations are allowed to transform the augmented matrices of Gauss:

- Any row of the augmented matrix can be multiplied by or divided by a constant (provided the constant does not equal zero).
- Any multiple of a row can be added to or subtracted from any other row.
- Any two rows of the augmented matrix can be interchanged.
Final Step (first strategy)

Change of y-coordinates:

\[ e_x x + \left( \frac{1}{2} e_x + \frac{7}{2} \sigma_y \right) y = \frac{7}{2} e_x + \frac{21}{2} \sigma_y \]

\[ e_y = \frac{1}{2} e_x + \frac{7}{2} \sigma_y \quad \Rightarrow \quad \sigma_y = -\frac{1}{7} e_x + \frac{2}{7} e_y \]

Final Gaussian Pauli vector equation:

\[ e_x x + e_y y = 2 e_x + 3 e_y \]
Augmented Matrix of the Final Step (first strategy)

The coefficients of the final Gaussian Pauli vector equation

\[(1 \ e_x + 0 \ e_y) \ x + (0 \ e_x + 1 \ e_y) \ y = 2 \ e_x + 3 \ e_y\]

are the elements of the final step of the augmented matrix:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
<td>7/2</td>
</tr>
<tr>
<td>0</td>
<td>7/2</td>
<td>21/2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

⇒ Solution values: \[x = 2\] \[y = 3\]

In some cases it is helpful to split steps into sub-steps to understand the strategy of Gauss.
Augmented Matrix of the Final Step (first strategy)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
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</tbody>
</table>

Multiply row one by $\frac{1}{4}$.

<table>
<thead>
<tr>
<th>$\frac{1}{2}$</th>
<th>$\frac{7}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Subtract row one from row two.

<table>
<thead>
<tr>
<th>$\frac{1}{2}$</th>
<th>$\frac{7}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{21}{2}$</td>
</tr>
</tbody>
</table>

Multiply row two by $\frac{2}{7}$.

<table>
<thead>
<tr>
<th>$\frac{1}{2}$</th>
<th>$\frac{7}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Subtract $\frac{1}{2}$ (row two) from row one.

<table>
<thead>
<tr>
<th>1</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
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</table>

$\Rightarrow$ $x = 2$

As intended, coefficient vector $b$ now represents a unit vector which lies parallel to the new y-coordinate axis:

$$e_y = b$$
Check of Result

System of two linear equations:

\[4 \times + 2 \times y = 14\]
\[x + 4 \times y = 14\]

Solution: \[x = 2\]
\[y = 3\]

Check of result:

\[4 \times 2 + 2 \times 3 = 8 + 6 = 14\]
\[2 + 4 \times 3 = 2 + 12 = 14\]

⇒ The result is correct.

The same results can be found by applying the Gaussian method in a different order (see following slides).
Second strategy

Now the y-coordinates are changed first:

\[(4 \sigma_x + \sigma_y) \cdot x + (2 \sigma_x + 4 \sigma_y) \cdot y = 14 \sigma_x + 14 \sigma_y\]

\[\sigma_y = -\frac{1}{2} \sigma_x + \frac{1}{4} e_y \iff e_y = 2 \sigma_x + 4 \sigma_y\]

New Gaussian Pauli vector equation:

\[
\left(\frac{7}{2} \sigma_x + \frac{1}{4} e_y\right) \cdot x + e_y \cdot y = 7 \sigma_x + \frac{7}{2} e_y
\]
Final Step (second strategy)

The \( x \)-coordinates are changed in the final step:

\[
\left( \frac{7}{2} \sigma_x + \frac{1}{4} e_y \right) x + e_y y = 7 \sigma_x + \frac{7}{2} e_y
\]

\[
e_x = \frac{7}{2} \sigma_x + \frac{1}{4} e_y \quad \Rightarrow \quad \sigma_x = \frac{2}{7} e_x - \frac{1}{14} e_y
\]

Final Gaussian Pauli vector equation:

\[
e_x x + e_y y = 2 e_x + 3 e_y
\]
Augmented Matrices  (second strategy)

The coefficients of the Pauli vector equations

\[
(4 \, \sigma_x + \sigma_y) \, x + (2 \, \sigma_x + 4 \, \sigma_y) \, y = 14 \, \sigma_x + 14 \, \sigma_y
\]

\[
\left(\frac{7}{2} \, \sigma_x + \frac{1}{4} \, e_y\right) \, x + (0 \, \sigma_x + 1 \, e_y) \, y = 7 \, \sigma_x + \frac{7}{2} \, e_y
\]

\[
(1 \, e_x + 0 \, e_y) \, x + (0 \, e_x + 1 \, e_y) \, y = 2 \, e_x + 3 \, e_y
\]

are the elements of the augmented matrices:

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<tr>
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<tbody>
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<td>x</td>
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<tr>
<td>0</td>
<td>1</td>
<td>3</td>
<td></td>
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</tbody>
</table>

Subtract \( \frac{1}{2} \) (row two) from row one.  
Multiply row two by \( \frac{1}{4} \).  
Multiply row one by \( \frac{2}{7} \).  
Subtract \( \frac{1}{14} \) (row one) from row two.  

\[ x = 2 \]
\[ y = 3 \]

An extended version with more sub-steps can be found at the following slide.
Extended Version of Augmented Matrices (second strategy)

<p>| | | |</p>
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<tbody>
<tr>
<td>x</td>
<td>y</td>
<td>d</td>
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<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

Multiply row two by $\frac{1}{4}$.

Subtract 2 times row two from row one.

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<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>1</td>
<td>(\frac{7}{2})</td>
</tr>
</tbody>
</table>

Multiply row one by $\frac{2}{7}$.

Subtract $\frac{1}{4}$ (row one) from row two.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>7/2</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>1</td>
<td>(\frac{7}{2})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>1</td>
<td>(\frac{7}{2})</td>
</tr>
</tbody>
</table>

$\Rightarrow \begin{align*}
    x &= 2 \\
    y &= 3
\end{align*}$

Of course the result is identical to the result of the first strategy.
Third Strategy

Now coefficient vector $\mathbf{a}$ will be chosen as unit vector which points into the direction of the new $y$-axis:

$$(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = 14 \sigma_x + 14 \sigma_y$$

$e_y = 4 \sigma_x + \sigma_y \quad \Rightarrow \quad \sigma_y = -4 \sigma_x + e_y$

New Gaussian Pauli vector equation:

$$e_y x + (-14 \sigma_x + 4 e_y) y = -42 \sigma_x + 14 e_y$$
Complete Diagram of Intermediate Step (third strategy)

After scaling the coordinate axes down, the complete diagram of the new equation \( e_y x + (-14 \sigma_x + 4 e_y) y = -42 \sigma_x + 14 e_y \) can be shown.
Final Step (third strategy)

And the new x-axis will point into the direction of coefficient vector \( \mathbf{b} \):

\[
\begin{align*}
\mathbf{e}_y \mathbf{x} + (-14 \sigma_x + 4 \mathbf{e}_y) \mathbf{y} &= -42 \sigma_x + 14 \mathbf{e}_y \\
\mathbf{e}_x &= -14 \sigma_x + 4 \mathbf{e}_y \\
\Rightarrow \quad \sigma_x &= -\frac{1}{14} \mathbf{e}_x + \frac{2}{7} \mathbf{e}_y
\end{align*}
\]

Final Gaussian Pauli vector equation:

\[
\mathbf{e}_y \mathbf{x} + \mathbf{e}_x \mathbf{y} = 3 \mathbf{e}_x + 2 \mathbf{e}_y
\]
Augmented Matrices (third strategy)

The coefficients of the Pauli vector equations

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = 14 \sigma_x + 14 \sigma_y\]

\[(0 \sigma_x + 1 e_y) x + (-14 \sigma_x + 4 e_y) y = -42 \sigma_x + 14 e_y\]

\[(0 e_x + 1 e_y) x + (1 e_x + 0 e_y) y = 3 e_x + 2 e_y\]

are the elements of the augmented matrices:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
<td>d</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>-14</td>
<td>-42</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Subtract 4 times row two from row one.

Divide row one by \((-14)\).

Add \(\frac{2}{7}\) (row one) to row two.

\(\Rightarrow\)  

\(y = 3\)

\(x = 2\)

An extended version with more sub-steps can be found at the following slide.
Extended Version of Augmented Matrices (third strategy)

\[
\begin{array}{ccc|c}
\hline
x & y & d \\
\hline
4 & 2 & 14 \\
1 & 4 & 14 \\
\hline
0 & -14 & -42 \\
1 & 4 & 14 \\
\hline
0 & 1 & 3 \\
1 & 4 & 14 \\
\hline
0 & 1 & 3 \\
1 & 0 & 2 \\
\end{array}
\]

Subtract 4 times row two from row one.

Divide row one by (−14).

Subtract 4 times row one from row two.

\[
\begin{array}{ccc|c}
\hline
x & y & d \\
\hline
0 & 1 & 3 \\
1 & 0 & 2 \\
\end{array}
\]

⇒ \( y = 3 \)

\( x = 2 \)

Again the result is identical to the results of previous strategies.
Fourth Strategy

Now coefficient vector $\mathbf{b}$ is chosen as unit vector which points into the direction of the new $x$-axis:

$$
(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = 14 \sigma_x + 14 \sigma_y
$$

$$
\sigma_x = \frac{1}{2} e_x - 2 \sigma_y \quad \iff \quad e_x = 2 \sigma_x + 4 \sigma_y
$$

New Gaussian Pauli vector equation:

$$
(2 e_x - 7 \sigma_y) x + e_x y = 7 e_x - 14 \sigma_y
$$
Final Step (fourth strategy)

And the new y-axis will point into the direction of coefficient vector $\mathbf{a}$:

$$(2 e_x - 7 \sigma_y) x + e_x y = 7 e_x - 14 \sigma_y$$

$$e_y = 2 e_x - 7 \sigma_y$$

$$\Rightarrow \quad \sigma_y = \frac{2}{7} e_x - \frac{1}{7} e_y$$

Final Gaussian Pauli vector equation:

$$e_y x + e_x y = 3 e_x + 2 e_y$$
Augmented Matrices (fourth strategy)

The coefficients of the Pauli vector equations

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = 14 \sigma_x + 14 \sigma_y\]

\[(2 e_x - 7 \sigma_y) x + (1 e_x + 0 \sigma_y) y = 7 e_x - 14 \sigma_y\]

\[(0 e_x + 1 e_y) x + (1 e_x + 0 e_y) y = 3 e_x + 2 e_y\]

are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>-7</td>
<td>0</td>
<td>-14</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

An extended version with more sub-steps can be found at the following slide.
Extended Version of Augmented Matrices (fourth strategy)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

Divide row one by 2.

Subtract 4 times row one from row two.

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

Divide row two by (–7).

Subtract 2 times row two from row one.

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

⇒ y = 3

x = 2

Again the result is identical to the results of previous strategies.
Inverse Matrices

As already discussed in previous lessons the solution of a system of linear equations can also be found with the help of the inverse matrix.

\[
\begin{align*}
4x + 2y &= 14 \\
x + 4y &= 14
\end{align*}
\]

\[
\Rightarrow \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix}
\]

Repetition: Definition of Inverse Matrices

Inverse matrices \( A^{-1} \) satisfy the relationship

\[
A^{-1} A = A A^{-1} = I
\]

The product of a matrix \( A \) and its inverse \( A^{-1} \) equals the identity matrix \( I \).

As already discussed this relationship can be used to solve systems of linear equations.
Repetition: Inverse Matrices

As already discussed in previous lessons the solution of a system of linear equations can also be found with the help of the inverse matrix.

\[ 4x + 2y = 14 \]
\[ x + 4y = 14 \]

\[ \Rightarrow \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix} \]

Multiplying \( A^{-1} \) from the left will result in:

\[ A^{-1}A \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 14 \\ 14 \end{bmatrix} \]

\[ \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 14 \\ 14 \end{bmatrix} \]

solution vector of the system of linear equations

The Gaussian method can be used to find the inverse of a matrix.
Finding Inverse Matrices with the Gaussian Method

If matrix $A$ is a $(n \times n)$-matrix, the definition of an inverse matrix $A A^{-1} = I$ can be split into $n$ different systems of $n$ different linear equations.

Thus in our $(2 \times 2)$-matrix example, we are able to split the defining equation

\[
\begin{array}{c|cc}
& x_1 & x_2 \\
& y_1 & y_2 \\
\hline 
4 & 2 & 1 \\
1 & 4 & 0 \\
\end{array}
\]

into the following two systems of two linear equations:

\[
\begin{array}{c|c}
& x_1 & x_2 \\
& y_1 & y_2 \\
\hline 
4 & 2 & 1 \\
1 & 4 & 0 \\
\end{array}
\]
Finding Inverse Matrices with the Gaussian Method

First (blue) system of linear equations:

\[ 4x + 2y = 1 \]
\[ x + 4y = 0 \]

Second (green) system of linear equations:

\[ 4x + 2y = 0 \]
\[ x + 4y = 1 \]

These two systems of two linear equations can be written as Pauli vector equations.

First (blue) system of linear equations:

\[ (4\sigma_x + \sigma_y)x + (2\sigma_x + 4\sigma_y)y = \sigma_x \]

Second (green) system of linear equations:

\[ (4\sigma_x + \sigma_y)x + (2\sigma_x + 4\sigma_y)y = \sigma_y \]

All row operations which will be carried out in the first (blue) system of linear equations are identical to the row operations which will be carried out in the second (green) system of linear equations.
Condensed Way of Writing Both Systems of Linear Equations

As all row operations which will be carried out in the first (blue) system of linear equations are identical to the row operations which will be carried out in the second (green) system of linear equations, we are able to write both equations

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_x\]

or\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_y\]

in a condensed way in one line only:

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_x \text{ or } \sigma_y\]

Of course, \(\sigma_x \neq \sigma_y\).

These are still two different equations.

These two equations can again be visualized in a diagram (see following slide).
The algebraic starting point is:

\[
\begin{align*}
\mathbf{a} \cdot x + \mathbf{b} \cdot y &= \sigma_x \perp \perp \sigma_y \\
(4 \sigma_x + \sigma_y) \cdot x + (2 \sigma_x + 4 \sigma_y) \cdot y &= \sigma_x \perp \perp \sigma_y
\end{align*}
\]
Transformation of Coordinates
(first strategy)

Change of $x$-coordinates:

$$(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_x \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} \sigma_y \end{pmatrix}$$

$e_x = 4 \sigma_x + \sigma_y \quad \Rightarrow \quad \sigma_x = \frac{1}{4} e_x - \frac{1}{4} \sigma_y$

$$1 e_x x + \left( \frac{1}{2} e_x + \frac{7}{2} \sigma_y \right) y = \frac{1}{4} e_x - \frac{1}{4} \sigma_y \begin{pmatrix} \sigma_y \end{pmatrix}$$

Change of $y$-coordinates:

$$1 e_x x + \left( \frac{1}{2} e_x + \frac{7}{2} \sigma_y \right) y = \frac{1}{4} e_x - \frac{1}{4} \sigma_y \begin{pmatrix} \sigma_y \end{pmatrix}$$

$e_y = \frac{1}{2} e_x + \frac{7}{2} \sigma_y \quad \Rightarrow \quad \sigma_y = -\frac{1}{7} e_x + \frac{2}{7} e_y$

$$1 e_x x + 1 e_y y = \frac{2}{7} e_x - \frac{1}{14} e_y \begin{pmatrix} \sigma_y \end{pmatrix}$$
Transformation of Coordinates
(second strategy)

Change of y-coordinates:

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_x \underbrace{\sigma_y}_{or}\]

\[e_y = 2 \sigma_x + 4 \sigma_y \implies \sigma_y = -\frac{1}{2} \sigma_x + \frac{1}{4} e_y\]

\[\left(\frac{7}{2} \sigma_x + \frac{1}{4} e_y\right) x + 1 e_y \quad y = \sigma_x \underbrace{\sigma_y}_{or} -\frac{1}{2} \sigma_x + \frac{1}{4} e_y\]

Change of x-coordinates:

\[\left(\frac{7}{2} \sigma_x + \frac{1}{4} e_y\right) x + 1 e_y \quad y = \sigma_x \underbrace{\sigma_y}_{or} -\frac{1}{2} \sigma_x + \frac{1}{4} e_y\]

\[e_x = \frac{7}{2} \sigma_x + \frac{1}{4} e_y \implies \sigma_x = \frac{2}{7} e_x - \frac{1}{14} e_y\]

\[1 e_x \quad x + 1 e_y \quad y = \underbrace{\frac{2}{7} e_x - \frac{1}{14} e_y}_{or} \underbrace{-\frac{1}{7} e_x + \frac{2}{7} e_y}_{or}\]
Diagram of the Final Situation
(first & second strategy)

The final algebraic equation equals:

\[ e_x \ x + e_y \ y = \frac{2}{7} e_x - \frac{1}{14} e_y \lor -\frac{1}{7} e_x + \frac{2}{7} e_y \]
Augmented Matrices (first strategy)

The coefficients of the Pauli vector equations

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_x \equiv \sigma_y\]

\[1 e_x x + \left(\frac{1}{2} e_x + \frac{7}{2} \sigma_y\right) y = \frac{1}{4} e_x - \frac{1}{4} \sigma_y \equiv \sigma_y\]

\[1 e_x x + 1 e_y y = \frac{2}{7} e_x - \frac{1}{14} e_y \equiv -\frac{1}{7} e_x + \frac{2}{7} e_y\]

are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

- Multiply row one by \(\frac{1}{4}\).
- Subtract \(\frac{1}{4}\) (row one) from row two.

\[
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{4} & 0 \\
0 & \frac{7}{2} & -\frac{1}{4} & 1
\end{bmatrix}
\]

- Subtract \(\frac{1}{7}\) (row two) from row one.
- Multiply row two by \(\frac{2}{7}\).

\[
\Rightarrow A^{-1} = \begin{bmatrix}
\frac{2}{7} & -\frac{1}{7} \\
\frac{1}{14} & \frac{2}{7}
\end{bmatrix} = \frac{1}{14} \begin{bmatrix}
4 & -2 \\
-1 & 4
\end{bmatrix}
\]
Extended Version of Augmented Matrices (first strategy)

\[ \begin{bmatrix} x & y & e_1 & e_2 \end{bmatrix} \]

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( \frac{7}{2} )</td>
<td>( -\frac{1}{4} )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( -\frac{1}{14} )</td>
<td>( \frac{2}{7} )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( \frac{2}{7} )</td>
<td>( -\frac{1}{7} )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( -\frac{1}{14} )</td>
<td>( \frac{2}{7} )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \Rightarrow A^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix} \]
Augmented Matrices (second strategy)

The coefficients of the Pauli vector equations

\[
(4 \sigma_x + \sigma_y) \ x + (2 \sigma_x + 4 \sigma_y) \ y = \sigma_x \ \text{or} \ \sigma_y
\]

\[
\left( \frac{7}{2} \sigma_x + \frac{1}{4} \ e_y \right) x + 1 \ e_y \quad y = \sigma_x \ \text{or} \ -\frac{1}{2} \ \sigma_x + \frac{1}{4} \ e_y
\]

\[
1 \ e_x \quad x + 1 \ e_y \quad y = \frac{2}{7} \ e_x - \frac{1}{14} \ e_y \ \text{or} \ -\frac{1}{7} \ e_x + \frac{2}{7} \ e_y
\]

are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>\frac{7}{2}</td>
<td>0</td>
<td>1</td>
<td>\frac{1}{2}</td>
</tr>
<tr>
<td>\frac{1}{4}</td>
<td>1</td>
<td>0</td>
<td>\frac{1}{4}</td>
</tr>
</tbody>
</table>

Subtract \(\frac{1}{2}\) (row two) from row one.

Multiply row two by \(\frac{1}{4}\).

Multiply row one by \(\frac{2}{7}\).

Subtract \(\frac{1}{14}\) (row one) from row two.

\[
\Rightarrow \ A^{-1} = \begin{bmatrix}
\frac{2}{7} & -\frac{1}{7} \\
\frac{1}{14} & \frac{2}{7}
\end{bmatrix} = \frac{1}{14} \begin{bmatrix}
4 & -2 \\
-1 & 4
\end{bmatrix}
\]
Extended Version of Augmented Matrices (second strategy)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Multiply row two by \( \frac{1}{4} \).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>\frac{1}{4}</td>
</tr>
</tbody>
</table>

Subtract 2 times row two from row one.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>-\frac{1}{2}</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>\frac{1}{4}</td>
</tr>
</tbody>
</table>

Multiply row one by \( \frac{2}{7} \).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>\frac{2}{7}</td>
<td>-\frac{1}{7}</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>\frac{1}{4}</td>
</tr>
</tbody>
</table>

Subtract \( \frac{1}{4} \) (row one) from row two.

\[
\Rightarrow A^{-1} = \begin{bmatrix}
\frac{2}{7} & -\frac{1}{7} \\
-\frac{1}{14} & \frac{2}{7}
\end{bmatrix} = \frac{1}{14} \begin{bmatrix}
4 & -2 \\
-1 & 4
\end{bmatrix}
\]
The final algebraic equation equals:

\[ e_y x + e_x y = -\frac{1}{14} e_x + \frac{2}{7} e_y \quad \text{or} \quad \frac{2}{7} e_x - \frac{1}{7} e_y \]

But the interpretation of this equation is not so easily found.
Augmented Matrices  (third strategy)

The coefficients of the Pauli vector equations

\[
(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_x \overline{=} \sigma_y
\]

\[
1 e_y x + (-14 \sigma_x + 4 e_y) y = \sigma_x \overline{=} -4 \sigma_x + e_y
\]

\[
1 e_y x + \quad 1 e_x y = -\frac{1}{14} e_x + \frac{2}{7} e_y \overline{=} \frac{2}{7} e_x - \frac{1}{7} e_y
\]

are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th></th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Subtract 4 times row two from row one.

<table>
<thead>
<tr>
<th></th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Divide row one by (-14).

<table>
<thead>
<tr>
<th></th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-\frac{1}{14}</td>
<td>\frac{2}{7}</td>
</tr>
<tr>
<td>1</td>
<td>\frac{2}{7}</td>
<td>-\frac{1}{7}</td>
</tr>
</tbody>
</table>

\[\Rightarrow \quad A^{-1} = \begin{bmatrix}
-\frac{1}{14} & \frac{2}{7} \\
\frac{2}{7} & -\frac{1}{7}
\end{bmatrix}\]

Matrix \(A^{-1}\) isn’t the inverse of matrix \(A\) yet:

\[A^{-1} \neq A^{-1}\]

A last step still is missing.
Augmented Matrices (third strategy)

Compared to the inverse $A^{-1}$, first and second row of matrix $A\uparrow^{-1}$ are interchanged. To change the x-direction into a new y-direction and the y-direction into a new x-direction at the same time, matrix $A^{-1}$ has to be multiplied with the permutation matrix from the left side:

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A\uparrow^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$

The complete table of augmented matrices then is:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>e₁</th>
<th>e₂</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-14</td>
<td>1</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Subtract 4 times row two from row one.

Divide row one by $(-14)$.

Add $\frac{2}{7}$ (row one) to row two.

Interchange row one and row two.

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix}$$
Extended Version of Augmented Matrices (third strategy)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>e₁</th>
<th>e₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>–14</td>
<td>1</td>
<td>–4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-\frac{1}{14} &amp; 2 \frac{7}{7} &amp; Subtract 4 times row one from row two.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
| 0  | 1  | -\frac{1}{14} & 2 \frac{7}{7} & \}
| 1  | 0  | \frac{2}{7} & 1 \frac{7}{7} & Interchange row one and row two. |
| 1  | 0  | \frac{2}{7} & 1 \frac{7}{7} |
| 0  | 1  | -\frac{1}{14} & 2 \frac{7}{7} |

\[ A^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix} \]
Augmented Matrices  (fourth strategy)

The coefficients of the Pauli vector equations

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = \sigma_x \tau_y \sigma_y\]

\[(2 e_x - 7 \sigma_y) x + 1 e_x y = \frac{1}{2} e_x - 2 \sigma_y \tau_y \sigma_y\]

\[1 e_y x + 1 e_x y = -\frac{1}{14} e_x + \frac{2}{7} e_y \tau_y \frac{2}{7} e_x - \frac{1}{7} e_y\]

are the elements of the augmented matrices:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
<td>e₁</td>
<td>e₂</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Divide row one by 2.
Subtract 2 times row one from row two.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>\frac{1}{2}</td>
<td>0</td>
</tr>
<tr>
<td>–7</td>
<td>0</td>
<td>–2</td>
<td>1</td>
</tr>
</tbody>
</table>

Add \(\frac{2}{7}\) (row two) to row one.
Divide row two by \((- 7)\).

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>–\frac{1}{14}</td>
<td>\frac{2}{7}</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>\frac{2}{7}</td>
<td>–\frac{1}{7}</td>
</tr>
</tbody>
</table>

Interchange row one and row two.

\[
\mathbf{A}^{-1} = \begin{bmatrix}
\frac{2}{7} & -\frac{1}{7} \\
\frac{1}{7} & -\frac{1}{7}
\end{bmatrix} = \frac{1}{14} \begin{bmatrix}
4 & -2 \\
-1 & 4
\end{bmatrix}
\]
Extended Version of Augmented Matrices (fourth strategy)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>(-7)</td>
<td>0</td>
<td>(-2)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( \frac{2}{7} )</td>
<td>(-\frac{1}{7})</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>(-\frac{1}{14})</td>
<td>(\frac{2}{7})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(\frac{2}{7})</td>
<td>(-\frac{1}{7})</td>
</tr>
</tbody>
</table>

Divide row one by 2.
Subtract 4 times row one from row two.
Divide row two by \((-7)\).
Subtract 2 times row two from row one.

Interchange row one and row two.

\[
\begin{bmatrix}
2 & 1 & \frac{1}{2} & 0 \\
2 & 1 & \frac{1}{2} & 0 \\
-7 & 0 & -2 & 1 \\
1 & 0 & \frac{2}{7} & -\frac{1}{7} \\
0 & 1 & -\frac{1}{14} & \frac{2}{7} \\
1 & 0 & \frac{2}{7} & -\frac{1}{7} \\
1 & 0 & \frac{2}{7} & -\frac{1}{7} \\
0 & 1 & -\frac{1}{14} & \frac{2}{7}
\end{bmatrix}
\]

\[
A^{-1} = \begin{bmatrix}
\frac{2}{7} & -\frac{1}{14} \\
-\frac{1}{14} & \frac{2}{7}
\end{bmatrix} = \frac{1}{14} \begin{bmatrix}
4 & -2 \\
-1 & 4
\end{bmatrix}
\]
Result

Original matrix: \( \mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \)

Inverse matrix: \( \mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{14} & \frac{2}{7} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix} \)

Check of the Inverse Matrix

To check the result, the original matrix \( \mathbf{A} \) should be multiplied by the inverse matrix \( \mathbf{A}^{-1} \) (or the inverse matrix \( \mathbf{A}^{-1} \) should be multiplied by the original matrix \( \mathbf{A} \)).

\[
\begin{bmatrix}
4 & 2 \\
1 & 4
\end{bmatrix}
\begin{bmatrix}
\frac{2}{7} & -\frac{1}{7} \\
-\frac{1}{14} & \frac{2}{7}
\end{bmatrix}
= \frac{1}{14}
\begin{bmatrix}
4 & -2 \\
-1 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{2}{7} & -\frac{1}{7} \\
-\frac{1}{14} & \frac{2}{7}
\end{bmatrix}
\begin{bmatrix}
4 & 2 \\
1 & 4
\end{bmatrix}
= \frac{1}{14}
\begin{bmatrix}
4 & -2 \\
-1 & 4
\end{bmatrix}
\]

\( \Rightarrow \) The inverse matrix is correct.
Solution of the System of Linear Equations

Now the solution of the system of two linear equations we took as an example can be found:

\[
\begin{align*}
4x + 2y &= 14 \\
x + 4y &= 14
\end{align*}
\]

\[
\Rightarrow \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 14 \\ 14 \end{bmatrix}
\]

solution vector of the system of linear equations

\[
\begin{array}{c|cc}
 & \multicolumn{2}{c}{14} \\
\hline
14 & 2 & -1 \\
-1 & 7 & 7
\end{array}
\]

\[
\Rightarrow \text{solution values: } \\
x = 2 \\
y = 3
\]
**Interpretation of the Inverse Matrix**

Original matrix: \( A = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \)

The original matrix can be interpreted as a set of two coefficient vectors:

\[
a = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \rightarrow \quad a = 4 \sigma_x + \sigma_y
\]
\[
b = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \rightarrow \quad b = 2 \sigma_x + 4 \sigma_y
\]

Inverse matrix: \( A^{-1} = \begin{bmatrix} \frac{2}{14} & \frac{-1}{14} \\ \frac{1}{14} & \frac{2}{14} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -2 \\ -1 & 4 \end{bmatrix} \)

The inverse matrix can be interpreted as a set of two other coefficient vectors:

\[
\sigma_x = \begin{bmatrix} 4/14 \\ -1/14 \end{bmatrix} \quad \rightarrow \quad \sigma_x = \frac{4}{14} a - \frac{1}{14} b
\]
\[
\sigma_y = \begin{bmatrix} -2/14 \\ 4/14 \end{bmatrix} \quad \rightarrow \quad \sigma_y = -\frac{2}{14} a + \frac{4}{14} b
\]
Interpretation of the Inverse Matrix

If we view the Gaussian method as a transformation of coordinates, the original matrix $A$ is equal to the set of new frame vectors

$$a = e_x \quad b = e_y$$

The elements of $A$ then are the coefficients of these new frame vectors* expressed as linear combinations of the base vectors $\sigma_x$, $\sigma_y$ of the old coordinate system.

And the inverse matrix $A^{-1}$ is equal to the set of old base vectors

$$\sigma_x \quad \sigma_y$$

The elements of $A^{-1}$ then are the coefficients of the old base vectors expressed as linear combinations of the frame vectors* $a$, $b$ of the new coordinate system.

* Base vectors usually are supposed to be orthonormal. As vectors $a = e_x$ and $b = e_y$ are not orthogonal, they are called frame vectors of the new coordinate system with oblique coordinate axes.
Finding the Lag Matrix

The Gaussian method of finding the inverse matrix can be generalized into a method of finding the lag matrix $B$ of a matrix product $AB = D$.

Example Problem

A firm manufactures two different types of final products $P_1$ and $P_2$. To produce these products the following quantities of two different raw materials $R_1$ and $R_2$ are required:

- 4 units of $R_1$ and 1 unit of $R_2$ are required to produce one unit of the first final product $P_1$.
- 2 units of $R_1$ and 4 units of $R_2$ are required to produce one unit of the second final product $P_2$.

In the first quarter 140 units of the first raw material $R_1$ and 140 units of the second raw material $R_2$ had been used up. In the second quarter 210 units of the first raw material $R_1$ and 350 units of the second raw material $R_2$ had been used up. In the third quarter 280 units of the first raw material $R_1$ and 70 units of the second raw material $R_2$ had been used up.

Find the production matrix which shows how many units of each final product were produced in the first, second, and third quarter.
Example Problem

Demand matrices \( \mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \) and
\( \mathbf{D} = \begin{bmatrix} 140 & 210 & 280 \\ 140 & 350 & 70 \end{bmatrix} \)
are given.

The production matrix \( \mathbf{B} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \) is unknown.

The matrix equation (see scheme of Falk)

\[
\begin{array}{c|ccc|ccc}
 & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\
\hline
4 & 2 & 140 & 210 & 280 \\
1 & 4 & 140 & 350 & 70 \\
\end{array}
\]

can now be split into three separate systems of linear equations:

\[
\begin{array}{c|c|c|c}
 & x_1 & x_2 & x_3 \\
\hline
y_1 & 4 & 2 & 140 \\
\end{array}
\begin{array}{c|c|c|c}
 & x_2 & y_2 & y_3 \\
\hline
y_2 & 4 & 2 & 210 \\
\end{array}
\begin{array}{c|c|c|c}
 & x_3 & y_3 \\
\hline
y_3 & 4 & 2 & 280 \\
\end{array}
\begin{array}{c|c|c|c}
 & x_1 & x_2 & x_3 \\
\hline
y_1 & 1 & 4 & 140 \\
\end{array}
\begin{array}{c|c|c|c}
 & x_2 & y_2 & y_3 \\
\hline
y_2 & 1 & 4 & 350 \\
\end{array}
\begin{array}{c|c|c|c}
 & x_3 & y_3 \\
\hline
y_3 & 1 & 4 & 70 \\
\end{array}
\]
Diagram of the Starting Point

We are now looking for a representation of the three vectors of constants:

\[ d_1 = 140 \sigma_x + 140 \sigma_y \]
\[ d_2 = 210 \sigma_x + 350 \sigma_y \]
\[ d_3 = 280 \sigma_x + 70 \sigma_y \]

as linear combinations of the coefficient vectors:

\[ a = 4 \sigma_x + \sigma_y \]
\[ b = 2 \sigma_x + 4 \sigma_y \]
As the third vector of constants $d_3 = 280 \sigma_x + 70 \sigma_y$ is parallel to coefficient vector $a = e_x = 4 \sigma_x + \sigma_y$, we immediately know that $y_3 = 0$. 
Transformation of Coordinates (first strategy)

The three systems of linear equations result in the following Pauli vector equations:

\[
(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = 140 \sigma_x + 140 \sigma_y \equiv 210 \sigma_x + 350 \sigma_y \equiv 280 \sigma_x + 70 \sigma_y
\]

\[
e_x = 4 \sigma_x + \sigma_y \Rightarrow \sigma_x = 0.25 e_x - 0.25 \sigma_y \Leftrightarrow \text{change of } x\text{-coordinates}
\]

\[
e_x x + (0.5 e_x + 3.5 \sigma_y) y = 35 e_x + 105 \sigma_y \equiv 52.5 e_x + 297.5 \sigma_y \equiv 70 e_x
\]

\[
e_y = 0.5 e_x + 3.5 \sigma_y \Rightarrow \sigma_y = -1/7 e_x + 2/7 e_y \Leftrightarrow \text{change of } y\text{-coordinates}
\]

\[
e_x x + e_y y = 20 e_x + 30 e_y \equiv 10 e_x + 85 e_y \equiv 70 e_x + 0 e_y
\]

\[\Rightarrow \text{Results: } x_1 = 20 \quad y_1 = 30 \quad x_2 = 10 \quad y_2 = 85 \quad x_3 = 70 \quad y_3 = 0
\]

\[\Rightarrow \text{20 units of the first final product } P_1 \text{ and 30 units of the second final product } P_2 \text{ were produced in the first quarter. 10 units of the first final product } P_1 \text{ and 85 units of the second final product } P_2 \text{ were produced in the second quarter. 70 units of the first final product } P_1 \text{ were produced in the third quarter.}
\]
Transformation of Coordinates (second strategy)

The three systems of linear equations result in the following Pauli vector equations:

\[(4 \sigma_x + \sigma_y) x + (2 \sigma_x + 4 \sigma_y) y = 140 \sigma_x + 140 \sigma_y \implies 210 \sigma_x + 350 \sigma_y \implies 280 \sigma_x + 70 \sigma_y\]

\[e_y = 2 \sigma_x + 4 \sigma_y \implies \sigma_y = -0.5 \sigma_x + 0.25 e_y \iff \text{change of y-coordinates}\]

\[(3.5 \sigma_x + 0.25 e_y) x + e_y y = 70 \sigma_x + 35 e_y \implies 35 \sigma_x + 87.5 e_y \implies 245 \sigma_x + 17.5 e_x\]

\[e_x = 3.5 \sigma_x + 0.25 e_y \implies \sigma_x = 2/7 e_x - 1/14 e_y \iff \text{change of y-coordinates}\]

\[e_x x + e_y y = 20 e_x + 30 e_y \implies 10 e_x + 85 e_y \implies 70 e_x + 0 e_y\]

\[\implies \text{Results: } x_1 = 20 \quad y_1 = 30 \quad x_2 = 10 \quad y_2 = 85 \quad x_3 = 70 \quad y_3 = 0\]

\[\implies \text{20 units of the first final product } P_1 \text{ and 30 units of the second final product } P_2 \text{ were produced in the first quarter. 10 units of the first final product } P_1 \text{ and 85 units of the second final product } P_2 \text{ were produced in the second quarter. 70 units of the first final product } P_1 \text{ were produced in the third quarter.}\]
### Augmented Matrices (first strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d₁</th>
<th>d₂</th>
<th>d₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>140</td>
<td>210</td>
<td>280</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>140</td>
<td>350</td>
<td>70</td>
</tr>
</tbody>
</table>

Multiply row one by $\frac{1}{4}$.

Subtract $\frac{1}{4}$ (row one) from row two.

<table>
<thead>
<tr>
<th>1</th>
<th>0.5</th>
<th>35</th>
<th>52.5</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.5</td>
<td>105</td>
<td>297.5</td>
<td>0</td>
</tr>
</tbody>
</table>

Subtract $\frac{1}{7}$ (row two) from row one.

Multiply row two by $\frac{2}{7}$.

$\Rightarrow B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}$

### Augmented Matrices (second strategy)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d₁</th>
<th>d₂</th>
<th>d₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>140</td>
<td>210</td>
<td>280</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>140</td>
<td>350</td>
<td>70</td>
</tr>
</tbody>
</table>

Subtract $\frac{1}{2}$ (row two) from row one.

Multiply row two by $\frac{1}{4}$.

<table>
<thead>
<tr>
<th>3.5</th>
<th>0</th>
<th>70</th>
<th>35</th>
<th>245</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1</td>
<td>35</td>
<td>87.5</td>
<td>17.5</td>
</tr>
</tbody>
</table>

Multiply row one by $\frac{2}{7}$.

Subtract $\frac{1}{14}$ (row one) from row two.

$\Rightarrow B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}$
### Augmented Matrices (third strategy)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>(d_1)</th>
<th>(d_2)</th>
<th>(d_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>140</td>
<td>210</td>
<td>280</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>140</td>
<td>350</td>
<td>70</td>
</tr>
<tr>
<td>0</td>
<td>−14</td>
<td>−420</td>
<td>−1190</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>140</td>
<td>350</td>
<td>70</td>
</tr>
</tbody>
</table>

**Subtract 4 times row two from row one.**

**Divide row one by (−14).**

**Add \(\frac{2}{7}\) (row one) to row two.**

\[
\begin{array}{c|cccc}
0 & 1 & 30 & 85 & 0 \\
1 & 0 & 20 & 10 & 70 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
1 & 0 & 20 & 10 & 70 \\
0 & 1 & 30 & 85 & 0 \\
\end{array}
\]

\[\Rightarrow B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}\]

### Augmented Matrices (fourth strategy)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>(d_1)</th>
<th>(d_2)</th>
<th>(d_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>140</td>
<td>210</td>
<td>280</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>140</td>
<td>350</td>
<td>70</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>70</td>
<td>105</td>
<td>140</td>
</tr>
<tr>
<td>−7</td>
<td>0</td>
<td>−140</td>
<td>−70</td>
<td>−490</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>30</td>
<td>85</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>20</td>
<td>10</td>
<td>70</td>
</tr>
</tbody>
</table>

**Divide row one by 2.**

**Subtract 2 times row one from row two.**

**Add \(\frac{2}{7}\) (row two) to row one.**

**Divide row two by (−7).**

\[
\begin{array}{c|cccc}
0 & 1 & 30 & 85 & 0 \\
1 & 0 & 20 & 10 & 70 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
1 & 0 & 20 & 10 & 70 \\
0 & 1 & 30 & 85 & 0 \\
\end{array}
\]

\[\Rightarrow B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix}\]
Result

If \( A = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \) and \( D = \begin{bmatrix} 140 & 210 & 280 \\ 140 & 350 & 70 \end{bmatrix} \)
are part of matrix equation \( A \, B = D \), then the unknown lag matrix will be \( B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix} \).

Check of Result

\[
\begin{array}{ccc}
20 & 10 & 70 \\
30 & 85 & 0 \\
\end{array}
\begin{array}{ccc}
4 & 2 & 140 \\
1 & 4 & 140 \\
\end{array}
\begin{array}{ccc}
210 & 280 & 70 \\
350 & 70 & \end{array}
\]

\( \Rightarrow \) The lag matrix is correct.

Finding the Lead Matrix

In the following problem the same values will be used to show how the lead matrix can be found.
Finding the Lead Matrix

Now matrix equation \( A B = D \) is given and matrices \( B \) and \( D \) are known, while matrix \( A \) has to be found.

Example Problem

A firm manufactures two different types of final products \( P_1 \) and \( P_2 \). To produce these products two different raw materials \( R_1 \) and \( R_2 \) are required.

In the first quarter 20 units of the first final product \( P_1 \) and 30 units of the second final product \( P_2 \) were produced.
In the second quarter 10 units of the first final product \( P_1 \) and 85 units of the second final product \( P_2 \) were produced.
In the third quarter 70 units of the first final product \( P_1 \) and no unit of the second final product \( P_2 \) were produced.

To produce these quantities of final products, the following quantities of raw materials had been used up:

In the first quarter 140 units of the first raw material \( R_1 \) and 140 units of the second raw material \( R_2 \) had been used up.
In the second quarter 210 units of the first raw material \( R_1 \) and 350 units of the second raw material \( R_2 \) had been used up.
In the third quarter 280 units of the first raw material \( R_1 \) and 70 units of the second raw material \( R_2 \) had been used up.

Find the demand matrix which shows how many units of each raw material \( R_1 \) and \( R_2 \) are required to produce one unit of each final product.
Finding the Lead Matrix

Now demand matrix \( A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \) is unknown, while the production matrix \( B = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix} \) and the matrix which shows the total demand of raw materials \( D = \begin{bmatrix} 140 & 210 & 280 \\ 140 & 350 & 70 \end{bmatrix} \) are given.

The unknown matrix \( A \) is the lead matrix of matrix product

\[ A B = D \]

But the solution strategies discussed in previous slides only show how a lag matrix can be found. Therefore the unknown lead matrix should be transformed into a lag matrix by transposing the matrix product.

\[ B^T A^T = D^T \]

Transposing a matrix product changes the of the matrices.
Transposition of Matrices

Transpose matrices can be found by converting columns of the original matrices into rows and rows of the original matrices into columns:

\[ \mathbf{B} = \begin{bmatrix} 20 & 10 & 70 \\ 30 & 85 & 0 \end{bmatrix} \quad \rightarrow \quad \mathbf{B}^T = \begin{bmatrix} 20 & 10 \\ 10 & 85 \\ 70 & 0 \end{bmatrix} \]

\[ \mathbf{D} = \begin{bmatrix} 140 & 210 & 280 \\ 140 & 350 & 70 \end{bmatrix} \quad \rightarrow \quad \mathbf{D}^T = \begin{bmatrix} 140 & 140 \\ 210 & 350 \\ 280 & 70 \end{bmatrix} \]

\[ \mathbf{A} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \quad \rightarrow \quad \mathbf{A}^T = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \]

Now the subscripts of the elements are in accordance with the column vector notation again.
Finding the Lead Matrix

The transposed matrix equation $B^T A^T = D^T$
(see scheme of Falk)

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$y_1$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
<td>140</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>210</td>
<td>350</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>280</td>
<td>70</td>
<td></td>
</tr>
</tbody>
</table>

can now be split into two separate systems of three linear equations:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>210</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>280</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$x_2$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>350</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>70</td>
</tr>
</tbody>
</table>
Pauli Vectors in Three-dimensional Space

Coefficient vectors and the two vectors of constants will be transferred again into Pauli vectors. But this time each system of linear equations consists of three linear equations.

Therefore the Pauli vectors will have three components, and three different base vectors $\sigma_x$, $\sigma_y$, and $\sigma_z$ are required.

$$ a = \begin{bmatrix} 20 \\ 10 \\ 70 \end{bmatrix} \quad \rightarrow \quad a = 20 \sigma_x + 10 \sigma_y + 70 \sigma_z $$

$$ b = \begin{bmatrix} 30 \\ 85 \\ 0 \end{bmatrix} \quad \rightarrow \quad b = 30 \sigma_x + 85 \sigma_y + 0 \sigma_z $$

$$ d_1 = \begin{bmatrix} 140 \\ 210 \\ 280 \end{bmatrix} \quad \rightarrow \quad d_1 = 140 \sigma_x + 210 \sigma_y + 280 \sigma_z $$

$$ d_2 = \begin{bmatrix} 140 \\ 350 \\ 70 \end{bmatrix} \quad \rightarrow \quad d_2 = 140 \sigma_x + 350 \sigma_y + 70 \sigma_z $$
Diagram of the Starting Point

The algebraic starting point is:

\[ a \begin{pmatrix} x \\ y \end{pmatrix} + b \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \]

or

\[ a \begin{pmatrix} x \\ y \end{pmatrix} + b \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 140 \sigma_x + 210 \sigma_y + 280 \sigma_z \\ 140 \sigma_x + 350 \sigma_y + 70 \sigma_z \end{pmatrix} \]
As we already know the solution of this system of linear equations is given by

\[ 4 \, a + 2 \, b = d_1 \]

and \[ 1 \, a + 4 \, b = d_2 \]

The solution can be found with appropriate transformations of the coordinates.
Again there are several different strategies to implement the transformation of coordinates:

The first new coordinate axis (and equally the second new axis) can be considered as a new x-axis or as a new y-axis or as a new z-axis.
Transformation of Coordinates (first strategy)

The two systems of linear equations result in the following Pauli vector equations:

\[(20 \sigma_x + 10 \sigma_y + 70 \sigma_z) x + (30 \sigma_x + 85 \sigma_y) y = 140 \sigma_x + 210 \sigma_y + 280 \sigma_z \]

\[\text{or} \quad 140 \sigma_x + 350 \sigma_y + 70 \sigma_z \]

Change of x-coordinates:

\[e_x = 20 \sigma_x + 10 \sigma_y + 70 \sigma_z \]

\[\Rightarrow \quad \sigma_x = 0.05 e_x - 0.5 \sigma_y - 3.5 \sigma_z \]

\[e_x x + (1.5 e_x + 70 \sigma_y - 105 \sigma_z) y = 7 e_x + 140 \sigma_y - 210 \sigma_z \]

\[\text{or} \quad 7 e_x + 280 \sigma_y - 420 \sigma_z \]

Change of y-coordinates:

\[e_y = 1.5 e_x + 70 \sigma_y - 105 \sigma_z \]

\[\Rightarrow \quad \sigma_y = -3/140 e_x + 1/70 e_y + 3/2 \sigma_y \]

\[e_x x + e_y y = 4 e_x + 2 e_y + 0 \sigma_z \]

\[\Rightarrow \quad \sigma_z = 1 e_x + 4 e_y + 0 \sigma_z \]

\[\Rightarrow \text{Results: } x_1 = 4 \quad y_1 = 2 \quad x_2 = 1 \quad y_2 = 4 \]

\[\Rightarrow 4 \text{ units of the first raw material } R_1 \text{ and } 1 \text{ unit of the second raw material } R_2 \text{ are required to produce one unit of the first final product } P_1. 1 \text{ unit of the first raw material } R_1 \text{ and } 4 \text{ units of the second raw material } R_2 \text{ are required to produce one unit of the second final product } P_2.\]
Transformation of Coordinates
(second strategy)

The two systems of linear equations result in the following Pauli vector equations:

\[
\begin{align*}
(20 \sigma_x + 10 \sigma_y + 70 \sigma_z) x + (30 \sigma_x + 85 \sigma_y) y &= 140 \sigma_x + 210 \sigma_y + 280 \sigma_z \\
-140 \sigma_x + 350 \sigma_y + 70 \sigma_z &\quad \text{or}
\end{align*}
\]

or

\[
(20 \sigma_x + 10 \sigma_y + 70 \sigma_z) x + (30 \sigma_x + 85 \sigma_y) y = 140 \sigma_x + 210 \sigma_y + 280 \sigma_z
\]

Change of \(y\)-coordinates:

\[
e_y = 20 \sigma_x + 10 \sigma_y + 70 \sigma_z
\]

\[
\Rightarrow \quad \sigma_y = -2 \sigma_x + 0.1 e_y - 7 \sigma_z
\]

\[
e_y x + (-140 \sigma_x + 8.5 e_y - 595 \sigma_z) y = -280 \sigma_x + 21 e_y - 1190 \sigma_z
\]

or

\[
-560 \sigma_x + 35 e_y - 2380 \sigma_z
\]

Change of \(z\)-coordinates:

\[
e_z = -140 \sigma_x + 8.5 e_y - 595 \sigma_z
\]

\[
\Rightarrow \quad \sigma_z = -4/17 \sigma_x + 1/70 e_y - 1/595 e_z
\]

\[
e_y x + e_z y = 0 \sigma_x + 4 e_y + 2 e_z = 0 \sigma_x + 1 e_y + 4 e_z
\]

\[
\Rightarrow \quad x_1 = 4 \quad y_1 = 2 \quad x_2 = 1 \quad y_2 = 4
\]

\[
\Rightarrow \quad 4 \text{ units of the first raw material } R_1 \text{ and } 1 \text{ unit of the second raw material } R_2 \text{ are required to produce one unit of the first final product } P_1. \quad 1 \text{ unit of the first raw material } R_1 \text{ and } 4 \text{ units of the second raw material } R_2 \text{ are required to produce one unit of the second final product } P_2.
\]
Transformation of Coordinates  
(third strategy)

The two systems of linear equations result in the following Pauli vector equations:

\[
\begin{align*}
(20 \sigma_x + 10 \sigma_y + 70 \sigma_z) x + (30 \sigma_x + 85 \sigma_y + 0 \sigma_z) y & = 140 \sigma_x + 210 \sigma_y + 280 \sigma_z \\
\text{or} & = 140 \sigma_x + 350 \sigma_y + 70 \sigma_z
\end{align*}
\]

Change of z-coordinates:

\[
e_z = 20 \sigma_x + 10 \sigma_y + 70 \sigma_z \quad \Rightarrow \quad \sigma_z = -\frac{2}{7} \sigma_x - \frac{1}{7} \sigma_y + \frac{1}{70} e_z
\]

\[
e_z x + (30 \sigma_x + 85 \sigma_y) y = 60 \sigma_x + 170 \sigma_y + 4 e_z \quad \text{or} \quad 120 \sigma_x + 340 \sigma_y + 1 e_z
\]

Change of x-coordinates:

\[
e_x = 30 \sigma_x + 85 \sigma_y \quad \Rightarrow \quad \sigma_x = \frac{1}{30} e_x - \frac{17}{6} \sigma_y
\]

\[
e_z x + e_x y = 2 e_x + 0 \sigma_y + 4 e_z \quad \text{or} \quad 4 e_x + 0 \sigma_y + 1 e_z
\]

\[
\Rightarrow \text{Results: } y_1 = 2 \quad x_1 = 4 \quad y_2 = 4 \quad x_2 = 1
\]

\[
\Rightarrow \text{4 units of the first raw material } R_1 \text{ and 1 unit of the second raw material } R_2 \text{ are required to produce one unit of the first final product } P_1. \text{ 1 unit of the first raw material } R_1 \text{ and 4 units of the second raw material } R_2 \text{ are required to produce one unit of the second final product } P_2.
\]
Augmented Matrices (first strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d₁</th>
<th>d₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
<td>140</td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>210</td>
<td>350</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>280</td>
<td>70</td>
</tr>
</tbody>
</table>

- Multiply row one by \( \frac{1}{20} \).
- Subtract \( \frac{1}{2} \) (row one) from row two.
- Subtract \( \frac{7}{2} \) (row one) from row three.
- Subtract \( \frac{3}{140} \) (row two) from row one.
- Multiply row two by \( \frac{1}{70} \).
- Add \( \frac{3}{2} \) (row two) to row three.

\[
A^T = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}
\]

⇒ As expected the unknown lead matrix equals:

\[
A = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}
\]
Augmented Matrices (second strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d₁</th>
<th>d₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
<td>140</td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>210</td>
<td>350</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>280</td>
<td>70</td>
</tr>
</tbody>
</table>

Subtract 2 times row two from row one.
Divide row two by 10.
Subtract 7 times row two from row three.

<table>
<thead>
<tr>
<th>0</th>
<th>-140</th>
<th>-280</th>
<th>-560</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.5</td>
<td>21</td>
<td>35</td>
</tr>
<tr>
<td>0</td>
<td>-595</td>
<td>-1190</td>
<td>-2380</td>
</tr>
</tbody>
</table>

Subtract \( \frac{4}{17} \) (row three) from row one.
Add \( \frac{1}{70} \) (row three) to row two.
Divide row one by \((-595)\).

<table>
<thead>
<tr>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ A^T = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix} \]

⇒ As expected the unknown lead matrix equals:

\[ A = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix} \]
Augmented Matrices  (third strategy)

The coefficients of the Pauli vector equations are the elements of the augmented matrices:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>d₁</th>
<th>d₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
<td>140</td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>210</td>
<td>350</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>280</td>
<td>70</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
<td>60</td>
<td>120</td>
</tr>
<tr>
<td>0</td>
<td>85</td>
<td>170</td>
<td>340</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Subtract $\frac{2}{7}$ (row three) from row one.

Subtract $\frac{1}{7}$ (row three) from row two.

Divide row three by 70.

Divide row one by 30.

Subtract $\frac{17}{6}$ (row one) from row two.

⇒ Second row of $A^T$

⇒ First row of $A^T$

⇒ As expected the unknown lead matrix equals:

\[
A = \begin{bmatrix}
4 & 2 \\
1 & 4
\end{bmatrix}
\]
Overconstrained Systems of Linear Equations

The example above showed how to solve a system of three linear equations with two variables. These systems, which have more equations than variables, are called overconstrained systems of linear equations.

The final augmented matrices of the examples possess a row which has only zeros as elements. This indicates that the mathematical information stored in these rows has not been required to find the solution.

We have been able to find the solution using the first two rows only (first strategy), using the last two rows only (second strategy), or using only the first and last rows (third strategy).
Inconsistent Systems of Linear Equations

Sometimes the equations of a system of linear equations are contradictory. Then no unique solution can be found. These systems of linear equations are called inconsistent.

As an example, the given equations of the problem discussed above are changed into:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$y_1$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
<td>140</td>
<td></td>
<td>20</td>
<td>30</td>
<td>140</td>
<td>140</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>210</td>
<td>350</td>
<td></td>
<td>10</td>
<td>85</td>
<td>210</td>
<td>350</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>280</td>
<td>70</td>
<td></td>
<td>70</td>
<td>0</td>
<td>285</td>
<td>68</td>
<td></td>
</tr>
</tbody>
</table>

consistent systems of linear equations

inconsistent systems of linear equations

The first two rows still result in the solution of the consistent systems, while the last row cannot be solved with these values.
Inconsistent Systems of Linear Equations

The inconsistency of linear equations is indicated by rows of the augmented matrices which have coefficient values of zero, but values of constants which differ from zero.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d₁</th>
<th>d₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>140</td>
<td>140</td>
</tr>
<tr>
<td>10</td>
<td>85</td>
<td>210</td>
<td>350</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>285</td>
<td>68</td>
</tr>
</tbody>
</table>

Multiply row one by \( \frac{1}{20} \).

Subtract \( \frac{1}{2} \) (row one) from row two.

Subtract \( \frac{7}{2} \) (row one) from row three.

\( x \) and \( y \) to row one.

Multiply row two by \( \frac{1}{70} \).

Add \( \frac{3}{2} \) (row two) to row three.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>d₁</th>
<th>d₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>70</td>
<td>140</td>
<td>280</td>
</tr>
<tr>
<td>0</td>
<td>–105</td>
<td>–205</td>
<td>–422</td>
</tr>
</tbody>
</table>

No real numbers \( x, y \) can solve these equations.

Multiples of zero can never be different from zero.